A Project Report

on

Study Of Blood Flow Models

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UNDERTAKING

I, Mainak Mandal (Reg No. 12MS092), BS-MS student IISER Kolkata, hereby declare that the project entitled "Study Of Blood Flow Models" submitted for the summer internship is done by me under the supervision of Dr. Manoj Kumar at Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad.

Signature of the Student:
Place:
Date:

1. Introduction

The understanding of blood flow dynamics is of immense importance in medical science as it helps to understand the relation between blood flow and vascular diseases and the change in flow characteristics under these circumstances. Some prosthetic or extra-corporeal flow devices like Haemo-dialyser which mimic and provide replacement for some body processes can be improved by study of blood flow. Exact mathematical description of blood flows can be quite complicated and almost impossible to solve but some simplified models can approximate the real situation to a great extent.

The layout of the report is as follows: Section 2 describes some basic structures an processes of human cardiovascular, Section 3 contains the detailed information about blood rheology and constituents of blood. Section 4 deals with some basic concepts of fluid dynamics. And lastly the mathematical models has been formulated and solved in section 5 with a concluding note in Section 6.

2. Cardiovascular System

The human blood vascular system consists of:

i. **The heart** - this organ has muscular and elastic walls which contract and relax rhythmically to maintain the oscillatory flow of blood through blood vessels.

ii. **The Arteries and arterioles** - help to distribute oxygenated blood throughout the body.

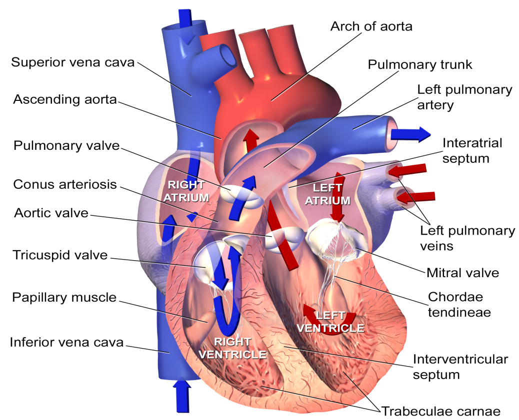
iii. **The Capillaries** - are in contact with cells, help to diffuse oxygen and nutrients to tissues.

iv. **The Veins and Venules** - which collect back deoxygenated blood from the body tissues.

Deoxygenated blood from the various parts of the body carrying metabolites enters the right atrium through venacava, from there it goes to the right ventricle. When the heart contracts, the tricuspid valve between the right atrium and the right ventricle closes and blood is pushed to the lungs through the pulmonary artery, which carries deoxygenated blood to the lungs where CO_2 is removed and blood is oxygenated. The oxygented blood return to the left atrium through the pulmonary vein. This is defined as the *pulmonary circulation*. Oxygenated blood from the left atrium goes to the left ventricle, from there due to contraction of the heart it is pushed into the aorta from which it travels to arteries and ultimately supplies oxygen and nutrition to different living tissues in the body. This is defined as *systemic circulation*. Apart from carrying O_2 and nutrients to living tissues and removing cellular wastes blood also maintains body pH and temperature and also helps to fight infections.

3. Physical properties of blood

3.1. Constituents of Blood



Sectional Anatomy of the Heart

Human blood is composed of plasma and formed elements. Formed elements comprise of RBCs , WBCs and platelets. Haemocrit is the percentage of blood occupied by formed elements. RBCs comprise of 99.9% of the formed elements in number per unit volume, rest .1% consists of WBCs and platelets.

i. **Plasma** - Straw colored fluid which contains significant amount of dissolved protein such as albumin, globulin and fibrinogen. It also contains regulatory proteins, electrolytes, organic nutrient and organic waste.

ii. **Red Blood Cells** - These are biconcave disc shaped cells red in color and have no nuclei at adult stage. $1\mu l$ of blood of a adult human contains about 4.5-6.3 billion RBCs. Their main function is to transport of respiratory gasses (O_2, CO_2). They have aa average life time of 120 days and are generated in the bone marrow.

iii. **White Blood Cells** - These are nucleated irregularly shaped cells mostly white in color. They are of two types, granulocytes and agranulocytes. $1\mu l$ of blood of a adult human contains about 6-9 thousand WBCs. They protect the body from pathogens and also remove toxins abnormal and dead cells. They are produced in bone marrow and also in lymph nodes.

iv. **Platelets** These are non nucleated flattened disk shaped cells, look round when viewed from above. They carry enzymes and other essential substances for blood clotting. The average life time of a platelet is around 9-12 days. These are also produced in bone marrow by magakaryocytes.

3.2. Blood Rheology

Blood is not a homogeneous fluid. It is a suspension of particles in plasma. Blood also does not behave as a Newtonian fluid under all conditions. There are many parameters which govern blood viscosity such as haemocrit, tube diameter, sheer rate etc.Plasma in isolation can be treated as a Newtonian fluid. Blood as a whole can also be treated as a Newtonian fluid at sufficiently high sheer rates(100 s^{-1}), which is the case in large arteries. Then the shearing stress and strain rate are related as follows:

$$\tau = \mu e \tag{1}$$

But in veins and other blood vessels where shear rate is smaller($10 s^{-1}$), blood tend to behave as a Casson Fluid. It means that blood has a certain yield stress, no flow of blood occurs below that yield stress. For small shear rates blood can be modeled by Casson's Equation.[3]

$$\begin{aligned} \sqrt{\tau} &= \sqrt{\mu e} + \sqrt{\tau_0} \quad (\tau \geq \tau_0) \\ e &= 0 \qquad\qquad\quad (\tau \leq \tau_0) \end{aligned} \tag{2}$$

Where τ is the shear stress, e is the shearing strain rate, τ_0 is a constant that is interpreted as yield stress and μ is a constant. Experimental data for low haemocrit($< .33$) show excellent fitting with this model, but higher haemocrit($> .39$) deviations are evident. The yield stress is very small though, experimentally it is found out to be in the order of .05 $dyne/cm^2$. But in these models blood is considered to be homogeneous. In large blood vessels blood can be approximated as homogeneous with negligible deviations, but in blood vessels with diameter comparable to the size of blood cells($8\mu m$)something strange occurs. Experimental data on apparent viscosity of blood flowing in narrow glass tubes shows a striking decrease as the tube diameter is reduced from 1 mm, a phenomenon known as *Fahraeus-Lindqvist effect.*[4] This phenomenon is caused by the tendency of blood cells to move to the center of the blood vessel in capillaries(dia.4-$10\mu m$). In this case a 2 layer model, a plasma layer(devoid of blood cells) near the vessel wall and a core layer consisting of blood cells with different viscous properties is to be considered.

4. Some basic concepts of fluid dynamics

Before we go into the details of the mathematical models for blood flow in human blood vessels, few basic concepts of fluid dynamics is discussed below:

4.1. Continuity equation for In-compressible Viscous fluid

Continuity equation describes the transport or flow of a conserved quantity. In this case it will be used to describe the flow of an in-compressible fluid. The differential form of the continuity equation for a q amount of quantity in volume $'V'$ is given by,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = \sigma. \tag{3}$$

where ρ is the amount of q per unit volume, t is time, $(\nabla \cdot)$ denotes the divergence, j is the flux of q and σ is the generation of q per unit volume per unit time. The physical explanation behind this equation is quite simple, the sum of the change in density of q per unit time and the divergence of flux of q is equal to the generation(or destruction) of q per unit time per unit volume. In fluid dynamics we have, $j = \rho u$ where v is the fluid velocity vector field and ρ is the fluid density. For in-compressible fluid ρ is constant so $(\partial \rho / \partial t) = 0$. If the volume V contains no sources or sinks then $\sigma = 0$. ρ being a constant we have,

$$\nabla \cdot v = 0 \tag{4}$$

If $v_x(x, y, z, t)$, $v_y(x, y, z, t)$,$v_z(x, y, z, t)$ and $p(x, y, z, t)$ denote the three velocity components respectively and the pressure at the point (x, y, z) at time t with viscosity coefficient μ then the mass-continuity equation for a viscous in-compressible fluid becomes,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{5}$$

4.2. Navier-Stokes Equation

Navier-Stokes equation is an important equation in fluid dynamics based on the Newton's 2nd law of motion, which is given by $\sum \vec{F} = m\vec{a}$. Considering a unit cube of fluid and considering the 2nd law in X-direction, we have $m\vec{a} = \rho(dv_x/dt)$. Since

v is a function of x, y, z and t so

$$\begin{aligned}\frac{dv_x}{dt} &= \frac{\partial v_x}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial v_x}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial v_x}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial v_x}{\partial z} \\ &= \frac{\partial v_x}{\partial t} + u\frac{\partial v_x}{\partial x} + v\frac{\partial v_x}{\partial y} + w\frac{\partial v_x}{\partial z}\end{aligned}$$

Now we have $m\vec{a} = \rho(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_z}{\partial z})$. For the other half of the Navier-stokes Equation we need to find the sum of various forces like forces due to pressure gradient, viscous forces and other external unbalanced forces. But the effect of gravity is not considered since we are not sure of its direction, latter we can introduce the gravity term into the equation. The force of gravity is given by $\rho g dV$. Pressure is a surface-stress[5] always acting normal to the surface of the control volume in a direction opposite to pressure gradient(direction of increasing pressure). So the force due to pressure gradient is given by, $F_p = -(\nabla \cdot p)dV$. For considering forces in one direction the right component of the gradient have to be used. Now the equation becomes

$$\rho(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}) = \sum F'_x - \frac{\partial p}{\partial x}$$

Another force in action is the viscous force. The force acting due shearing stress is given by the $(\nabla \cdot \tau)dV$. Unlike pressure which is a vector and has only one force couples per direction, shearing stress is a tensor quantity which has three force couples per direction.[5] Including the viscous forces in the equation we get

$$\rho(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}) = \sum F''_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Now if we consider a Newtonian fluid where the shearing stress is always proportional to the local shear rate, that is $\tau = \mu e$ at every point in the fluid. And strain is given by $e = -\partial v/\partial x$, so

$$\begin{aligned}\rho(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}) &= \sum F''_x - \frac{\partial p}{\partial x} + \mu(\frac{\partial e_{xx}}{\partial x} + \frac{\partial e_{yx}}{\partial y} + \frac{\partial e_{zx}}{\partial z}) \\ &= \sum F''_x - \frac{\partial p}{\partial x} + \mu(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2})\end{aligned}$$

The Navier-stokes equation for a viscous in-compressible Newtonian fluid is given by

$$\rho(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z})=\sum F''_x-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}) \tag{6}$$

$$\rho(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z})=\sum F''_y-\frac{\partial p}{\partial y}+\mu(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}) \tag{7}$$

$$\rho(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z})=\sum F''_z-\frac{\partial p}{\partial z}+\mu(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}) \tag{8}$$

Solving these coupled partial differential equations using initial conditions and boundary conditions we can find the velocity vector field. These are complicated enough so cannot be solved analytically, only can be approximately solved using numerical methods. We can simplify (6),(7) and (8) if the net external forces are set to zero and consider the equations in 2-dimensions

$$\rho(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y})=-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial x^2}) \tag{9}$$

$$\rho(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y})=-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial x^2}) \tag{10}$$

4.3. Poiseuille flow

While picturing blood flowing through blood vessels the simplest situation that we can think of is a fluid flowing steadily through a straight uniform rigid tube. So the velocity has only one component, in the axial direction. Since the problem has cylindrical symmetry we will consider the equations (5),(9) and (10) in cylindrical polar coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial}{\partial z}(rv_z)=0 \tag{11}$$

$$\rho(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z})=-\frac{\partial p}{\partial r}+\mu(\frac{\partial^2 v_r}{\partial r^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}) \tag{12}$$

$$\rho(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z})=-\frac{\partial p}{\partial z}+\mu(\frac{\partial^2 v_z}{\partial r^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}) \tag{13}$$

Now since velocity only has a non zero component in the z direction so v_θ and v_r are zero. All velocity components are time independent. Therefore the mass continuity equation gives

$$\frac{\partial v_z}{\partial z} = 0$$
$$\Rightarrow v_z = v(r) \tag{14}$$

and (12) and (13) gives,

$$\frac{\partial p}{\partial r} = 0 \tag{15}$$
$$\mu(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}) - \frac{\partial p}{\partial z} = 0 \tag{16}$$

From (15) we know that p is a function of z only and from (16) we can conclude that $-(\partial p/\partial z)$ is a constant. Let that constant pressure gradient be denoted by p', then (16) becomes,

$$\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr} + \frac{p'}{\mu} = 0$$
$$\Rightarrow \frac{1}{r}\frac{d}{dr}(\frac{dv_r}{dr}) + \frac{p'}{\mu} = 0$$
$$\Rightarrow \int d(r\frac{dv_z}{dr}) = -\frac{p'}{\mu}\int rdr$$

integrating we get,

$$r\frac{dv_z}{dr} = -\frac{p'}{2\mu}r^2 + \ C$$
$$\Rightarrow v_z(r) = -\frac{p'}{4\mu}r^2 + C\ ln(r) + C_1 \tag{17}$$

Now putting in the boundary condition that velocity on the axis must be finite, we have $C = 0$ and from the no slip condition($v(R) = 0$)we have $C_1 = (p'/4\mu)R^2$ so,

$$v_z(r) = \frac{p'}{4\mu}(R^2 - r^2) \tag{18}$$

The above velocity field is parabolic with the velocity at the axis being maximaum and the velocity at the walls being zero. We can also calculate the flux from the expression of the velocity. Flux Q is given by,

$$Q = \int_0^a 2\pi r\ v(r)dr = \frac{\pi p'}{8\mu}R^4 \tag{19}$$

This can be used to experimentally determine μ.

4.4. Reynolds Number

Reynolds number is a dimensionless number which roughly determines whether the flow will be turbulent or laminar. It is the ratio of inertial forces to viscous forces. In equations (6),(7) and (8) the terms on the L.H.S denotes the inertial forces and the R.H.S contains the viscous forces. So the Inertial forces are of the order $\rho v^2 L^{-1}$ and the viscous forces are of the order $\mu v L^{-2}$. So dimensionally

$$Re = \frac{\rho v^2 L^{-1}}{\mu v L^{-2}} = \frac{\rho v L}{\mu} = 1$$

So Reynolds number is dimensionless and can be use to predict the type of flow. Typical flows in straight pipes having Reynolds number below 2000 are laminar. When Reynolds number is large then the inertial forces are large as compared to the viscous forces, which tend to diffuse the fluid causing turbulence. But in case of blood, the flows remain laminar even at Reynolds number as high as 5000 or more.

4.5. Inlet length and plug flow

The velocity profile for a fully developed Poiseuille flow in a straight tube is parabolic. But before the fluid enters the tube then the velocity profile is straight and parallel to the axis of the tube. The fluid must pass though a finite length in the tube to attain the parabolic velocity profile. As soon as the fluid enters the tube the fluid layer next to the tube wall will be forced to have zero velocity and in turn it will apply shearing force in the next layer. this will tend to form the velocity gradient in the radial direction, the flow will tend to achieve the parabolic velocity profile asymptotically. Entrance length is defined as the length of the tube in which 99 % of the final velocity profile is achieved.[2] The flow in the entry length consists of two parts, the region near the wall which is called the boundary layer and the layer near the center called the core flow or plug flow.[6] It can be proven that the entrance length is proportional to the Reynolds number.

4.6. Bernoulli Equation

Bernoulli's equation can be called an form of the energy conservation principle. The equation states that the sum of the pressure energy, kinetic energy and potential energy in a fixed volume of fluid flow remains conserved. Mathematically,

$$p + \rho g z + \frac{1}{2}\rho v^2 = constant \tag{20}$$

This is particularly helpful when the cross-section of the blood vessel changes, so the pressure and velocity change can be found out through out this equation.

Stenososed Artery

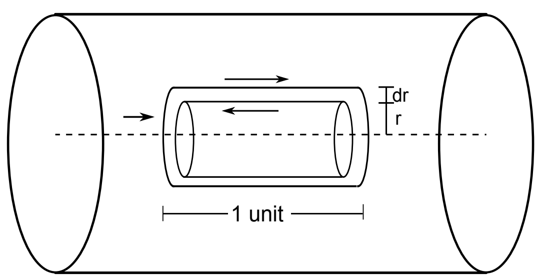
Stenosis is narrowing of the arterial lumen due to plaque deposition or other types of abnormal tissue development. Since the flux remains constant while flowing through a constriction so $A_1 v_1 = A_2 v_2$. Since the cross-section decreases so the velocity has to increase in a constriction hence the kinetic energy increases causing the pressure to fall, which in turn causes the stenosis to grow.

Aneurysm

An aneurysm is caused by the weakening of arterial wall where a bulge occurs and the cross-section of the blood vessel increases considerably. Due to increase in cross-section the velocity hence the kinetic energy falls, causing a pressure rise. This further causes the bulge to increase which might ultimately cause the blood vessel to burst.

5. Blood Flow models

5.1. Steady non-Newtonian flows in cylindrical tubes



We consider a laminar flow of a non-Newtonian fluid flowing through a cylindrical tube under a constant pressure gradient p'. The control volume is bounded by two cylinders of unit length and of radius r and $r + dr$. Due to the pressure gradient the forward force on the control volume is $p' \times 2\pi r \times dr$. The shearing stress, $\tau(r)$ produces a force on the surfaces of the control volume. The force on the inner surface $F(r) = 2\pi r\tau$ since the length of the cylinder is one unit. Now the force on the outer cylinder is,

$$\begin{aligned}F(r+dr) &= F(r) + F'(r)\cdot \Delta r \\ &= 2\pi r\tau + 2\pi\frac{d}{dr}(r\tau)dr\end{aligned}$$

balancing force on the axial direction we have,

$$\begin{aligned}&2\pi\frac{d}{dr}(\tau r)\cdot dr = p'2\pi r\cdot dr \\ \Rightarrow &\int d(\tau r) = \int p'r\cdot dr \\ \Rightarrow &\tau r = \frac{1}{2}p'r^2 + C \\ \Rightarrow &\tau = \frac{1}{2}p'r + \frac{C}{r}\end{aligned}$$

The stress is finite at $r = 0$ therefore $C = 0$. So,

$$\tau = \frac{1}{2}p'r \tag{21}$$

We know that strain $e = -(dv/dr)$. For a non-Newtonian fluid $\tau = f(e)$ so we have,

$$\frac{1}{2}p'r = f(-\frac{dv}{dr}) \tag{22}$$

Integrating (22) subjected to the boundary conditions that $v = v$ at $r = 0$ and $v = 0$ at $r = R$ we can have v as a function of r. Once we have v as a function of r we can calculate the flux.

$$j = \int_0^R 2\pi r\cdot v(r)dr$$

Integrating by parts we have,

$$\begin{aligned}j &= 2\pi\left\{\left[\frac{r^2}{2}v\right]_0^R - \int_0^R \frac{r^2}{2}\frac{dv}{dr}dr\right\} \\ &= \pi\int_0^R r^2(-\frac{dv}{dr})dr \\ &= \pi\int_0^R r^2 e\cdot dr\end{aligned}$$

5.1.1. Steady flow for Herschel-Bulkley fluid in a cylindrical tube

For a Herschel-Bulky fluid the sheering stress and the strain rate are related as:

$$\tau = \tau_0 + \mu e^n \qquad (\tau \geq \tau_0)$$
$$e = 0 \qquad (\tau \leq \tau_0)$$

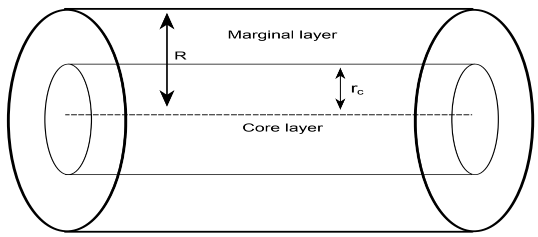


Figure 1. 2 layer model

In this model we have two layers, marginal layer close to the wall and a core layer. In the core region $\tau \leq \tau_o$ so the strain rate is zero so the velocity gradient in the radial direction is zero. So the core layer flows as a plug, with a constant velocity profile. Let the radius of the core is r_c and the radius of the tube is R. At the surface of the plug the stress is τ_0.We consider the control volume to be a cylinder of unit length and of radius r_c. Equation the forces on the control volume in the axial direction we have:

$$p' \cdot \pi r_c^2 = \tau_0 \cdot 2\pi r_c$$
$$\Rightarrow \tau_0 = \frac{1}{2} p' r_c$$

In the non core region, the equation (22) becomes:

$$\frac{1}{2} p' r = \mu e^n + \tau_0$$
$$\Rightarrow e = \left[\frac{(1/2)p'r - \tau_0}{\mu}\right]^{1/n}$$
$$= \left[\frac{p'}{2\mu}\right]^{1/n} (r - r_c)^{1/n}$$
$$\Rightarrow \frac{dv}{dr} = -\left[\frac{p'}{2\mu}\right]^{1/n} (r - r_c)^{1/n}$$

integrating we have:

$$\Rightarrow \ -\int_v^0 dv = \left[\frac{p'}{2\mu}\right]^{1/n} \int_r^R (r - r_c)^{1/n} dr$$

$$\Rightarrow \ v = \frac{n}{n+1}\left[\frac{p'}{2\mu}\right]^{1/n} \left[(R - r_c)^{n+1/n} - (r - r_c)^{n+1/n}\right] \tag{23}$$

Using boundary condition $v = v_c$ at $r = r_c$ in (23), where v_c is the velocity of the core layer:

$$v_c = \frac{n}{n+1}\left[\frac{p'}{2\mu}\right]^{1/n} \left[(R - r_c)^{n+1/n}\right]$$

$$v_c = \frac{n}{n+1}\left[\frac{p'}{2\mu}\right]^{1/n} R^{1+\frac{1}{n}} \left[(1 - \frac{r_c}{R})^{n+1/n}\right] \tag{24}$$

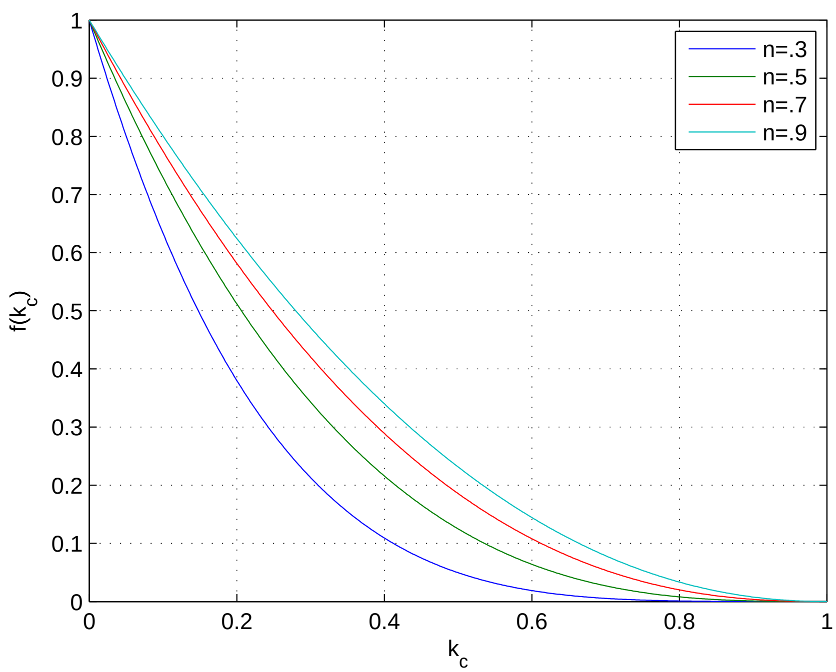


Figure 2. Relative variation of v_c with k_c for various n

Now let, $\frac{r_c}{R} = k_c$. From (24) we obtain:

$$v_c = \frac{n}{n+1}\left[\frac{p'}{2\mu}\right]^{1/n} R^{1+\frac{1}{n}} \left[(1 - k_c)^{n+1/n}\right] \tag{25}$$

Relative variation of v_c with k_c for various n is plotted in Fig (2). From the graph we can see that the velocity of the core layer v_c decreases with increasing r_c and τ_0. Total flux through the tube is given by:

$$
\begin{aligned}
j &= \pi r_c^2 v_c + \int_{r_c}^{R} 2\pi r v dr \\
&= \pi r_c^2 v_c + 2\pi \frac{n}{n+1} \left[\frac{p'}{2\mu}\right]^{1/n} \int_{r_c}^{R} \left[r(R-r_c)^{n+1/n} - r(r-r_c)^{n+1/n} \right] dr
\end{aligned}
$$

integrating by parts we obtain:

$$
\begin{aligned}
=&\pi r_c^2 v_c + 2\pi \frac{n}{n+1} \left[\frac{p'}{2\mu}\right]^{1/n} (R-r_c)^{1+1/n} \frac{1}{2} \left[R^2 - r_c^2\right] \\
&- 2\pi \frac{n}{n+1} \left[\frac{p'}{2\mu}\right]^{1/n} \left[\frac{n}{2n+1} (r-r_c)^{\frac{1}{n}+2} - \frac{n}{(2n+1)} \int (r-r_c)^{\frac{1}{n}+2} dr \right]_{r_c}^{R} \\
=&\pi \frac{n}{n+1} r_c^2 \left[\frac{p'}{2\mu}\right]^{1/n} \left[(R-r_c)^{1+1/n}\right] + \pi \frac{n}{n+1} \left[\frac{p'}{2\mu}\right]^{1/n} (R-r_c)^{1+1/n} \left[R^2 - r_c^2\right] \\
&- 2\pi \frac{n}{n+1} \left[\frac{p'}{2\mu}\right]^{1/n} \left[\frac{n}{2n+1} (r-r_c)^{\frac{1}{n}+2} - \frac{n^2}{(2n+1)(3n+1)} (R-r_c)^{\frac{1}{n}+3} \right] \\
=&\pi \frac{n}{n+1} \Bigg[r_c^2 (R-r_c)^{1+1/n} + (R^2 - r_c^2)(R-r_c)^{1+1/n} \\
&- \frac{2n}{2n+1} R (R-r_c)^{2+1/n} + \frac{2n^2}{(2n+1)(3n+1)} (R-r_c)^{3+1/n} \Bigg] \\
=&\pi \frac{n}{n+1} R^{3+1/n} \Bigg[\left(\frac{r_c}{R}\right)^2 \left(1 - \frac{r_c}{R}\right)^{1+1/n} + \left\{ 1 - \left(\frac{r_c}{R}\right)^2 \right\} \left(1 - \frac{r_c}{R}\right)^{1+1/n} \\
&- \frac{2n}{2n+1} \left(1 - \frac{r_c}{R}\right)^{2+1/n} + \frac{2n^2}{(2n+1)(3n+1)} \left(1 - \frac{r_c}{R}\right)^{3+1/n} \Bigg] \qquad (26)
\end{aligned}
$$

now let $k_c = \frac{r_c}{R}$, now from eqn (26) we obtain,

$$
\begin{aligned}
j =&\pi \frac{n}{n+1} \left(\frac{p'}{2\mu}\right) R^{3+1/n} \Bigg[(k_c)^2 (1-k_c)^{1+1/n} + (1+k_c)(1-k_c)^{2+1/n} \\
&- \frac{2n}{2n+1} (1-k_c)^{2+1/n} + \frac{2n^2}{(2n+1)(3n+1)} (1-k_c)^{3+1/n} \Bigg] \\
=&\pi \frac{n}{n+1} \left(\frac{p'}{2\mu}\right) R^{3+1/n} f(k_c) \qquad (27)
\end{aligned}
$$

So the flux for steady flow of a Herschel-Bulkley fluid through a circular cylinder is given by the equation (27). In Fig (5.1.1.) the relative variation of j with k_c for different n is shown. Its obvious from the plot that, with the increase k_c i.e. increase of yield stress the total flux decreases in general for different n.

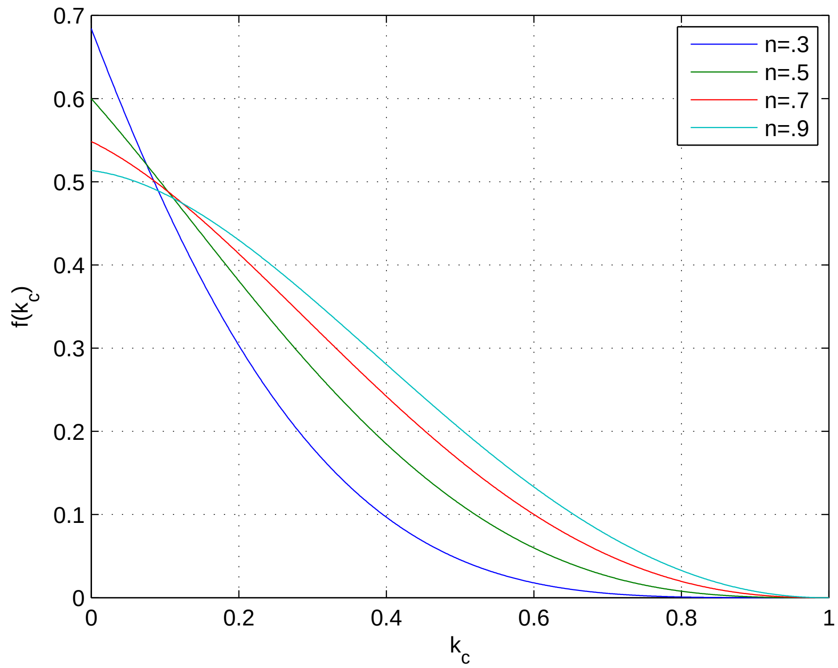


Figure 3. Relative variation of j with k_c for different n

It also means that flux decreases with increasing radius of core layer, though the core layer has maximum velocity. This is because with increasing τ_0 velocity of the core layer also decreases.

5.1.2. Steady flow for Casson fluid in a cylindrical tube

For a Casson fluid we have $\tau^{1/2} = \tau_0^{1/2} + (\mu e)^{1/2}$. Rearranging we have:

$$\Rightarrow \; e = \frac{\left(\sqrt{\frac{p'r}{2}} - \sqrt{\frac{p'r_c}{2}}\right)^2}{\mu}$$

$$\Rightarrow -\frac{dv}{dr} = \frac{p'}{2\mu}\left(\sqrt{r} - \sqrt{r_c}\right)^2$$

integrating we have:

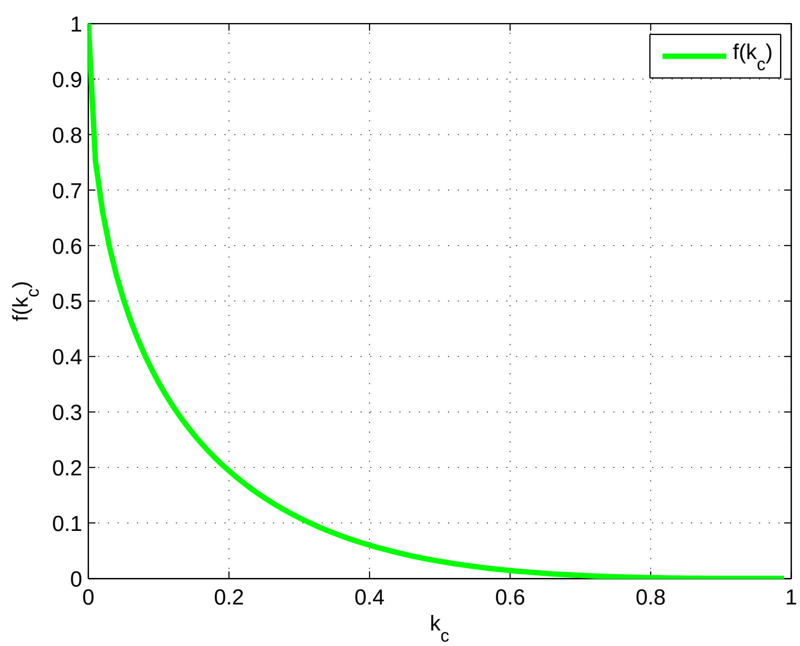
$$
\begin{aligned}
-\int_{v}^{0} dv &= \frac{p'}{2\mu}\int_{r}^{R}\left(r + r_c - 2\sqrt{r \cdot r_c}\right) dr \\
\Rightarrow\ v &= \frac{p'}{2\mu}\left[\frac{r^2}{2} + r \cdot r_c - \frac{4}{3} r^{3/2} r_c^{1/2}\right]_r^R \\
\Rightarrow\ v &= \frac{p'}{2\mu}\left[\frac{R^2}{2} - \frac{r^2}{2} + R \cdot r_c - r \cdot r_c + \frac{4}{3} R^{3/2} r_c^{1/2} - \frac{4}{3} r^{3/2} r_c^{1/2}\right] \qquad (28)
\end{aligned}
$$

using boundary condition, $v = v_c$ when $r = r_c$:

$$
\begin{aligned}
v_c &= \frac{p'}{2\mu}\left[\frac{R^2}{2} - \frac{4}{3} R^{3/2} r_c^{1/2} + R \cdot r_c - \frac{1}{6} r_c^2\right] \\
&= \frac{p' R^2}{4\mu}\left[1 + 2\frac{r_c}{R} - \frac{8}{3}\left(\frac{r_c}{R}\right)^{1/2} - \frac{1}{3}\left(\frac{r_c}{R}\right)^2\right] \qquad (29)
\end{aligned}
$$

let $\frac{r_c}{R} = k_c$ therefore, from equation (29) we have:

$$
\begin{aligned}
v_c &= \frac{p' R^2}{4\mu}\left[1 + 2k_c - \frac{8}{3} k_c^{1/2} - \frac{1}{3} k_c^2\right] \\
&= \frac{p' R^2}{4\mu} \cdot f(k_c)
\end{aligned}
$$



We know that $k_c = \frac{r_c}{R} = \frac{2\tau_0}{p'R}$, therefore for $\tau_0 = 0$ we have $[v_c]_{\tau_0=0} = \frac{p'R^2}{4\mu}$

$$\frac{v_c}{[v_c]_{\tau_0=0}} = f(k_c)$$

From the figure we can see the relative variation of v_c with k_c. The graph shows that the plug velocity decreases till k_c reaches 0.6 then it it remains almost steady with increasing k_c. The flux through the cylindrical tube is given by:

$$\begin{aligned}
j &= \pi r_c^2 v_c + \pi \frac{p'}{\mu} \int_{r_c}^{R} r \left[\frac{R^2}{2} - \frac{r^2}{2} + \frac{4}{3} R^{\frac{3}{2}} r_c^{\frac{1}{2}} - \frac{4}{3} r^{\frac{3}{2}} r_c^{\frac{1}{2}} + R r_c - r r_c \right] \\
&= \pi r_c^2 v_c + \pi \frac{p'}{\mu} \int_{r_c}^{R} r \left[\frac{rR^2}{2} - \frac{r^3}{2} + \frac{4}{3} R^{\frac{3}{2}} r_c^{\frac{1}{2}} r - \frac{4}{3} r^{\frac{5}{2}} r_c^{\frac{1}{2}} + R r_c r - r^2 r_c \right] \\
&= \pi r_c^2 v_c + \pi \frac{p'}{\mu} \left[\frac{r^2 R^2}{4} - \frac{r^4}{8} + \frac{2}{3} R^{\frac{3}{2}} r_c^{\frac{1}{2}} r^2 - \frac{8}{21} r^{\frac{7}{2}} r_c^{\frac{1}{2}} + \frac{1}{2} R r^2 r_c - \frac{1}{3} r^3 r_c \right]_{r_c}^{R} \\
&= \pi r_c^2 v_c + \pi \frac{p'}{\mu} \Bigg[-\frac{1}{8}(R^4 - r_c^4) - \frac{8}{21} r_c^{\frac{1}{2}} (R^{\frac{7}{2}} - r_c^{\frac{7}{2}}) - \frac{1}{3} r_c (R^3 - r_c^3) + \\
&\quad \frac{2}{3} R^{\frac{3}{2}} r_c^{1/2} (R^2 - r_c^2) + \frac{1}{2} R r_c (R^2 - r_c^2) + \frac{1}{4} R^2 (R^2 - r_c^2) \Bigg] \qquad (30)
\end{aligned}$$

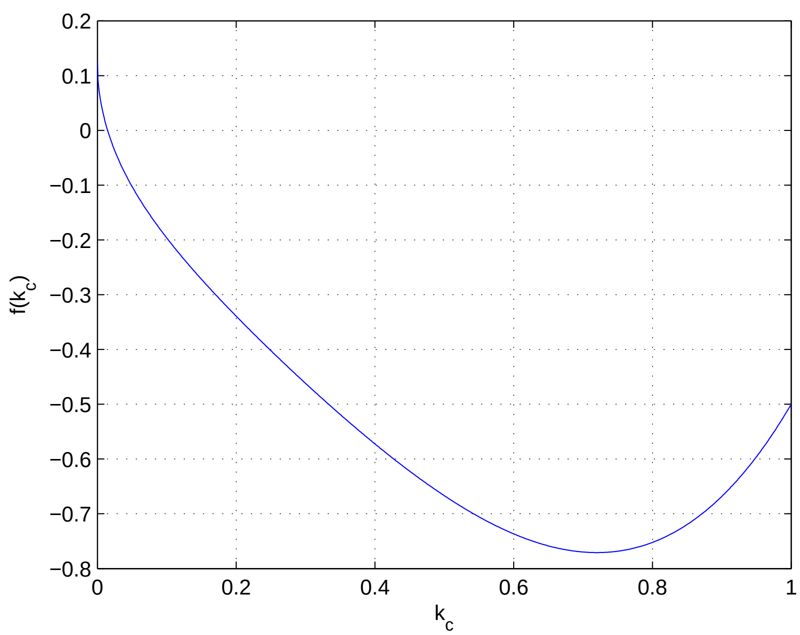


Figure 4. 2 layer model

Using the previous substitution of $k_c = \frac{r_c}{R}$, we obtain:

$$j = \pi\frac{p'R^4}{4\mu}k_c^2\left[1 + 2k_c - \frac{8}{3}k_c^{1/2} - \frac{1}{3}k_c^2\right] + \pi\frac{p'}{\mu}R^4\Bigg[-\frac{1}{8}(1-k_c^4) - \frac{8}{21}k_c^{\frac{1}{2}}(1-k_c^{\frac{7}{2}})$$
$$-\frac{1}{3}k_c(1-k_c^3) - \frac{2}{3}k_c^{\frac{1}{2}}(1-k_c^2) + \frac{1}{2}k_c(1-k_c^2) + \frac{1}{4}(1-k_c^2)\Bigg] \quad (31)$$
$$= \pi\frac{p'R^4}{4\mu}f(k_c) \quad (32)$$

This is the expression for flux for steady flow through a cylindrical tube for a Casson fluid. From the plot of j vs k_c it can be observed that the flux decreases till k_c reaches 0.7 then it increases slightly. This is in good agreement with the variation of v_c with k_c.

6. Conclusion

In this study we have modeled the blood flow blood vessels. In this model we have considered blood to be Newtonian when the strain strain rate is large, which is the case in large arteries. The non-Newtonian properties of blood also has been considered for low strain rates blood, when blood flows through narrow arteries. Blood flow through narrow blood vessels is conciderd to be bi-layered. Casson fluid and Herschel-Bulkley fluid model have been used to study this bi-layered model of blood. Casson fluid model can be used to explain velocity profile for moderate strain rates which occur in arterioles of diameter around $.1mm$. But velocity profiles in even smaller arterioles whose diameter are less than $0.065mm$ deviate significantly from the Casson fluid model, but can still be explained by Herschel-Bulkley fluid model.[7] Thus in this study we have used both of the fluid models in appropriate situations to get a fuller picture. The model discussed here can be made more realistic by considering other characteristics of blood flow like pulsatile nature of blood flow, visco-elastic properties of blood, elastic nature of blood vessels and their unusual curvature.

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